CERTAIN FEATURES OF A FIELD OF INTERNAL WAVES GENERATED BY A LOCAL SOURCE OF PERTURBATIONS IN A FLOW OF A TWO-LAYER FLUID

V. E. Veden'kov and V. F. Sannikov

UDC 551.466.81

The study [1] described a method of calculating an exact (in a linear formulation) solution of the problem of steady-state three-dimensional waves generated by a local source of perturbations in a stratified flow. Sample calculations were performed for a fluid with a constant Weisel—Brant frequency using the Bussinesq approximation and the criterion of a "solid cover" on the upper boundary of the fluid. The investigation conducted in [1] showed that the difference of the exact solution from well-known approximate solutions [2, 3] is significant in the neighborhoods of the leading edges of individual modes when the flow velocity exceeds the velocity of the waves of the corresponding modes. The asymptotic solutions constructed in [4] agree with the results in [1]. In this study, the methods in [1, 4] are used to calculate internal waves in a two-layer fluid. A simple model of a fluid with a steep density curve makes it possible to determine the role of the wavelength dependence of the vertical profiles of internal waves in the formation of a field of generated waves. A two-layer stratification model has been used repeatedly in the problem in question to study both steady [5-7] and nonsteady [8, 9] waves. However, only [7] contains examples of calculations of the amplitudes, and these calculations were performed with approximate asymptotic formulas.

Here, we will examine a flow of an inviscid incompressible fluid which is borderless in the horizontal directions. The fluid flows with a constant velocity c in the positive direction of the x axis and consists of two layers: the density and thickness of the top layer are ρ_1 and H_1; the density and thickness of the bottom layer are ρ_2 and H_2. Meanwhile, $\rho_1 < \rho_2$. Let a wave generator be a point source of the intensity Q. The generator is located a distance H_3 from the interface. In a linear formulation, the wave motions of the fluid created by the source are described by the Poisson equation for the potential of the perturbed velocities

$$\Delta\varphi = Q\delta(x, y, z - H_3) \quad (-H_2 < z < H_1,\ z \neq 0) \tag{1}$$

with the boundary conditions

$$L\varphi = 0 \ (z = H_1), \quad \partial\varphi/\partial z = 0 \ (z = -H_2), \tag{2}$$

to which we must add the radiation condition — the absence of wave disturbances upflow — and the condition of continuity of the functions $\partial\varphi/\partial z$ and $\rho_0(z)L\varphi$ on the undisturbed interface between the layers (z = 0). Here, $L = c^2\partial^2/\partial x^2 + g\partial/\partial z$; g is acceleration due to gravity; $\rho_0(z) = \rho_1$ at $z > 0$ and $\rho_0(z) = \rho_2$ at $z < 0$; $\delta(\cdot)$ is the delta function.

Following [1], we find an expression for the vertical component of the perturbed velocities $w = \partial\varphi/\partial z$ in the form of the sum of single integrals:

$$\begin{gathered}
w(x, y, z) = \frac{Q}{2\pi}\rho_0(H_3)\left\{w_{01} + w_{11} + \sum_{n=0}^{\infty} w_{n2}\right\}, \\
w_{n1} = \int_{-\pi/2}^{\pi/2} F_n(\theta; R, \omega, z)\, d\theta, \\
F_n(\theta; R, \omega, z) = -H\left(r_n^2\right) H\left[\cos(\theta - \omega)\right] W_n(z;\theta)\, W_{nz}(H_3;\theta)\sin(R\Delta_n), \\
w_{n2} = \pi^{-1}\int_{-\pi/2}^{\pi/2} W_n(z;\theta)\, W_{nz}(H_3;\theta)\left[H\left(r_n^2\right) G_1(R\Delta_n) + H\left(-r_n^2\right) G_i(R\Delta_n)\right] d\theta, \\
G_1(u) = \int_0^{\infty} t\,(t^2+1)^{-1}\exp(-|u|\,t)\, dt, \quad G_i(u) = \int_0^{\infty} t\,(t^2+1)^{-1}\exp(iut)\, dt.
\end{gathered} \tag{3}$$

Sevastopol'. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 84-89, March-April, 1987. Original article submitted February 17, 1986.

 0021-8944/87/2802-0242$12.50

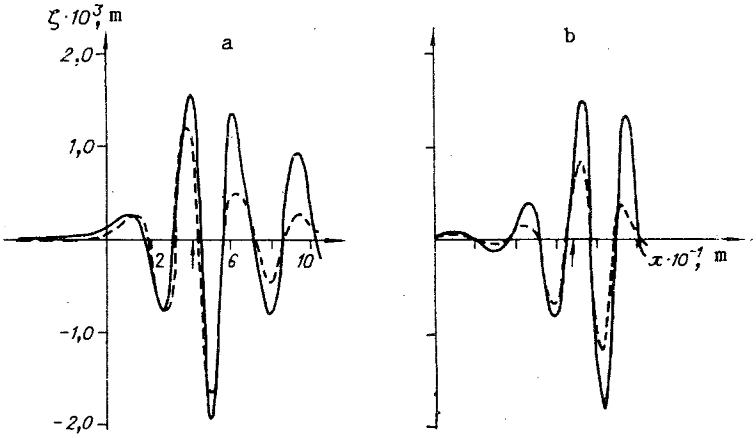

Fig. 1

Here, R and ω are polar coordinates of the horizontal plane (x, y); $\Delta_n = |r_n| \cos(\theta - \omega)$; the functions $G_1(u)$ and $G_i(u)$ are expressed [1] through integral exponential functions; $H(\cdot)$ is the Heaviside unit function; $\beta_n = r_n^2(\theta)$ and $W_n(z;\ \theta)$ are eigenvalues ($\beta_0 > \beta_1 > \ldots$) and orthonormalized eigenfunctions $\left(\int\limits_{-H_2}^{H_1} \rho_0(z) W_n W_m dz = \delta_n^m,\right.$ δ_n^m is the Kronecker symbol) of the homogeneous boundary-value problem

$$W_{zz} - \beta W = 0 \ (-H_2 < z < H_1, \ z \neq 0),$$
$$W_z - \varkappa W = 0 \ (z = H_1), \ W = 0 \ (z = -H_2); \tag{4}$$

W and $\rho_0(W_z - \kappa W)$ are continuous at the point z = 0; $\kappa = g(c \cos\theta)^{-2}$. Problem (4) has a denumerable number of real eigenvalues, these eigenvalues being solutions of the dispersion relation

$$\beta(1 + \gamma \operatorname{th} rH_1 \operatorname{th} rH_2) - \varkappa r(\operatorname{th} rH_1 + \operatorname{th} rH_2) +$$
$$+ \varkappa^2 \varepsilon \operatorname{th} rH_1 \operatorname{th} rH_2 = 0, \ r = \beta^{1/2}, \ \gamma = \rho_1/\rho_2, \ \varepsilon = 1 - \gamma. \tag{5}$$

At $n \geq 2$, the eigenvalues $\beta_n(\theta) < 0$ at all θ.

In Eq. (3), the term w_{01} describes the contribution of the surface mode to the wave field, while w_{11} describes the contribution of the internal mode. The remaining terms of the two-layer model are nonwave terms, and they can be ignored at large distances from the wave generator because $w_{n2} = O(r^{-2})$ at $R \to \infty$ ($n \geq 2$) [10]. This estimate is also valid for the integrals w_{n2} (n = 0, 1) in those cases when the flow velocity $c < c_n$. Here, $c_n = \sqrt{g/\kappa_n}$. The quantities κ_0 and κ_1 ($\kappa_0 < \kappa_1$) are solutions of Eq. (5) at β = 0. If $c > c_n$ (n = 0, 1), then the contribution of the term w_{n2} in the neighborhood of the boundary of the wave region $\omega = \arcsin(c_n/c)$ if $O(R^{-2/3})$ [1]. Features of the contribution of the integral w_{n1} (n = 0, 1) were described in [1]. We note only that in earlier studies, expressions for the field w(x, y, z) which were asymptotic at $R \to \infty$ were represented as the sum of only these terms and their subsequent estimates obtained by the stationary-phase method.

To check the model, we compared theoretical calculations with data from a laboratory experiment [11]. The solid lines in Fig. 1 show the theoretical values of wave amplitude on the interface of an ovoid with a midsection radius $R_m = 0.01$ and a ratio $L/R_m = 12$. The rest of the parameters had the following values: $H_1 = 0.09$, $H_2 = 0.3$, $H_3 = 0.02$ m, c = 0.223 m/sec, γ = 0.8, *a* − y = 0.15 m, b − y = 0.25 m. In the theoretical calculations, the ovoid was modeled by a source–sink system. The parameters of this system were determined just as for an infinite uniform liquid [12]. The expression for the displacement ξ was derived from (3) by means of the kinematic relation $c\partial\xi/\partial x = w$. Analysis of Fig. 1 shows satisfactory agreement between the theory and experiment. The increase in the deviation of the corresponding curves with an increase in x and y is evidently due to the fact that the theoretical model did not consider the transience of the wave generation and dissipation and the fact that an idealized scheme was used for the wave generator.

Equations (3) make it possible to calculate the entire region of the wave field. However, it is more economical to use asymptotic solutions at large distances from the wave generator.

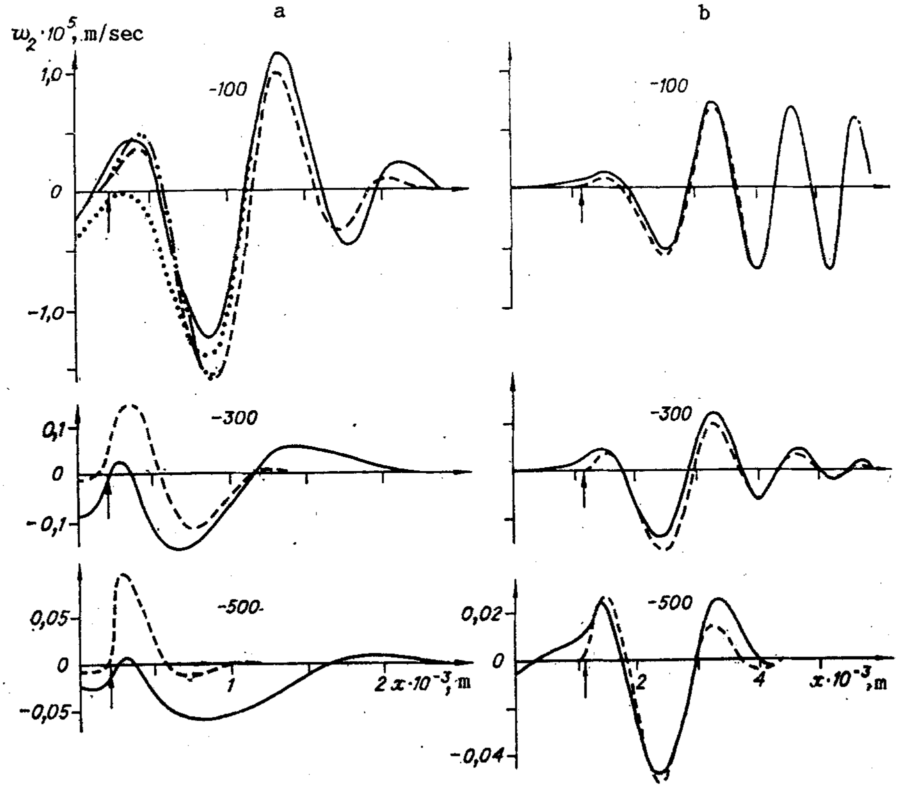

Fig. 2

In the case $c > c_n$, the principal disturbances inside the wave region are described by the integral w_{n1} [1]. The principal term of the asymptote of this integral is determined by the contribution of the stationary point [9]. In the neighborhood of the leading edge, the terms w_{n1} and w_{n2} are of the same order of magnitude [1], and the asymptote of the complete solution (3) is expressed through the Airy function [4]. It can be shown that at $c > c_n$, the uniform estimate of the contribution of the n-th mode (n = 0.1) has the form

$$w_n = -Q\rho_0(H_3)R^{-2/3}A_i'(-BR^{2/3})\,W_n(z;\theta)\times$$
$$\times W_{nz}(H_3;\theta)\Big/\sqrt{-2B^{1/2}\Delta''_{n\theta\theta}} + O(R^{-4/3}), \tag{6}$$
$$B = r_n^2(\theta)\left[3\sin(\theta-\omega)/dr_n^2/d\theta\right]^{2/3},$$

where the right sides are calculated at the stationary point of the solutions of the equation $\Delta_{n\theta}{}' = 0$. At $BR^{2/3} \gg 1$, replacement of the derivative of the Airy function in (6) by its asymptotic expression allows (6) to be changed to a form which coincides completely with the asymptotic estimate obtained by the stationary-phase method. Figures 2 and 3 show examples of calculation of the contribution of the internal mode according to exact formulas (3) (solid lines) and asymptotic formula (6) (dashed lines). The dotted and dot–dash lines show the contribution of the term w_{11} and its asymptote according to the stationary-phase method. The calculations were performed with H_1 = 100 m, H_2 = 3900 m, c = 3 m/sec, Q = 1 sec^{-1}, γ = 0.996; for Fig. 2, z = 0 m, $a - y$ = 200 m, $b - y$ = 1000 m. The values of H_3 (m) are indicated next to the curves; H_3 = −50 m, y = 1000 m for Fig. 3, and values of z (m) are given around the lines. It must be noted that inside the wave region at sufficiently large distances R, the stationary-phase method gives better estimates of the characteristics of the generated waves than do the single integrals w_{n1}.

Analysis of Figs. 2 and 3 shows that the range of application of the asymptotic formulas depends considerably on the parameters z and H_3. With an increase in the depth of immersion of the source (Fig. 2), significant wave disturbances tend to be located closer to the leading edge, indicated by the arrow. In this case, there is an increase in the difference between the exact and asymptotic solutions. The same result is obtained with increasing distance from the interface (Fig. 3) when the position of the source is fixed.

This effect is connected with the dependence of the amplitude factor $\Phi_n(z, H_3, \theta) = W_n(z;\theta)W_{nz}(H_3;\theta)$ on the wave angle θ. This dependence is exponential in character, and in

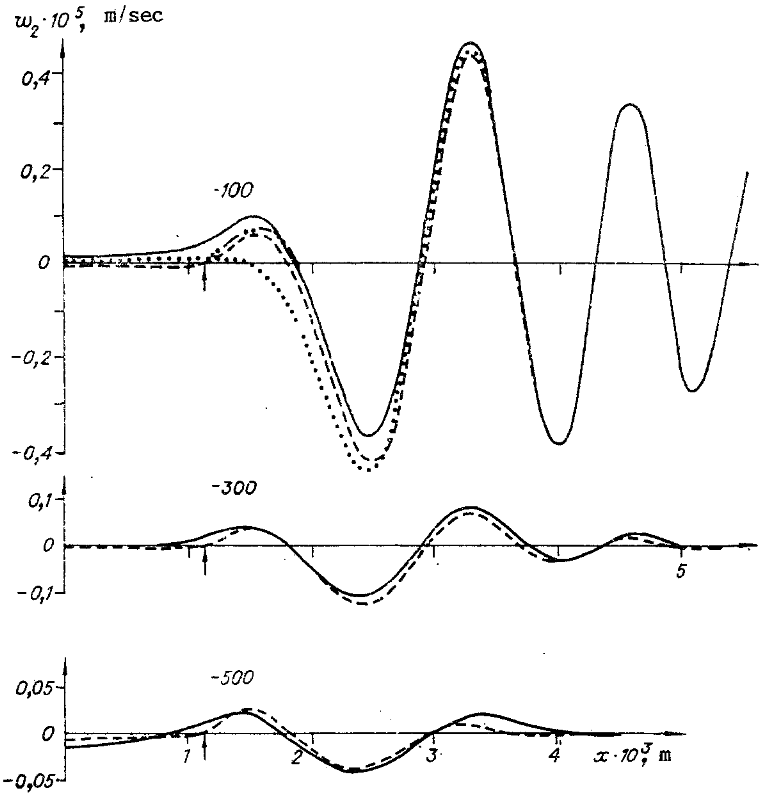

Fig. 3

the bottom layer of fluid $\Phi_n(z, H_3; \theta) \sim \exp[r_n(\theta)(z + H_3)]$ at $-r_n(\theta)(z - H_3) \gg 1$. It is known [1] that $r_n(\theta) \to \infty$ at $|\theta| \to \pi/2$. Considering this, it might be possible to construct a more accurate asymptotic estimate (3) if we use the descent approach. The use of this approach is attended by serious obstacles, however, since it requires the solution of the equation $(d/d\theta)\{r_n(0)[a + H_3 + iR\cos(\theta - \omega)]\} = 0$ in the complex region of values of θ. Let $\delta = |z - H_3|/R \ll 1$ and $\theta = \theta_0$ is a simple stationary point ($d\Delta_n/d\theta = 0$ at $\theta = \theta_0$) when $\delta = 0$. Then the phase S_n of the asymptotic estimate has the form

$$S_n = R\left\{\Delta_n(\theta_0) + \frac{[\delta r'_{n\theta}(\theta_0)]^2}{2\Delta''_{n\theta\theta}(\theta_0)}\right\} - \pi/4 + o(\delta^2).$$

It follows from here in particular that an estimate obtained by the descent method gives larger values of wavelength $[\Delta''_{n\theta\theta}(\theta_0) < 0]$ than the stationary-phase method. This observation is consistent with the results of numerical calculations shown in Figs. 2 and 3.

LITERATURE CITED

1. V. F. Sannikov, "Nearby field of steady waves generated by a local source of disturbances in a flow of stratified fluid," in: Theoretical Investigations of Wave Processes in the Ocean [in Russian], Sebastopol (1983).
2. V. F. Sannikov and L. V. Cherkesov, "Development of three-dimensional internal waves in a flow of stratified fluid," in: Surface and Internal Waves [in Russian], Sebastopol (1981).
3. I. V. Sgurova and V. A. Sukharev, "Generation of internal waves by local disturbances in a fluid with a specified change in density through the depth," Izv. Akad. Nauk SSSR Fiz. Atmos. Okeana, 17, No. 6 (1981).
4. V. A. Borovikov, Yu. V. Vladimirov, and M. Ya. Kel'bert, "Field of internal gravity waves excited by localized sources," Izv. Akad. Nauk SSSR Fiz. Atmos. Okeana, 20, No. 6 (1984).
5. A. A. Hudimac, "Ship waves in a stratified ocean," J. Fluid Mech., 11, No. 2 (1961).
6. G. D. Crapper, "Ship waves in a stratified ocean," J. Fluid Mech., 29, No. 4 (1967).
7. L. D. Filimonova and L. V. Cherkesov, "Internal waves generated by moving disturbances," in: Morsk. Gidrofiz. Issled., No. 1 (1977).
8. V. S. Fedosenko and L. V. Cherkesov, "Development of ship waves in a nonuniform liquid," Izv. Akad. Nauk SSSR Mekh. Zhidkh. Gaza, No. 4 (1970).

9. V. S. Fedosenko and L. V. Cherkesov, "Nonsteady waves in a nonuniform liquid of finite depth," Morak. Gidrofiz. Issled., No. 2 (1972).
10. V. F. Sannikov, "Steady internal waves generated by a local source of disturbances in a flow," in: Modeling Surface and Internal Waves [in Russian], Sebastopol (1984).
11. V. I. Bukreev and A. V. Gusev, "Waves generated by the movement of a sphere and an ovoid in a two-layer fluid," Transactions of the All-Union Conference "Wave Processes in Seas and Oceans," Sebastopol, 1983. Moscow (1984), Submitted to VINITI 9.01.84, No. 281-84.
12. N. J. Kotchin, L. A. Kibel, and N. W. Rose, Theoretical Fluid Mechanics [Russian translation], Vol. 1, GITTL, Moscow (1955).

CAUCHY INTERNAL WAVE SCATTERING BY DENSITY FIELD INHOMOGENEITIES

S. P. Budanov, A. S. Tibilov, and V. A. Yakovlev

UDC 532.593

The stationary problem of internal wave (IW) scattering by density field inhomogeneities was considered in a linear formulation in [1] in an unbounded medium with a constant Brunt–Väisäilä frequency. The important role is shown for this mechanism in the IW energy redistribution between different modes. Domains are defined in which the scattered IW amplitude is substantially different from zero. The corresponding nonstationary problem is discussed in this paper.

Let IW characterized by the density $\rho_\Phi(r, t)$ and velocity $U_\Phi(r, t)$ fields exist in a medium. At the time t = 0 local "mixing" (spoilage of the ρ_Φ and U_Φ field distributions) of the medium occur in a domain of space F_1. Neglecting rotation of the earth and the viscosity forces in a Boussinesq approximation, this nonstationary problem has the form

$$L_U\{\rho, \mathbf{U}\} = Q(\mathbf{U}), \quad \mathbf{U}|_{t=0} = \begin{cases}\mathbf{U}_\Phi, & \mathbf{r} \notin D_1,\\ \mathbf{U}_i, & \mathbf{r} \in D_1,\end{cases}$$
$$L_\rho\{\rho, \mathbf{U}\} = \varphi(\rho, \mathbf{U}), \quad \rho|_{t=0} = \begin{cases}\rho_\Phi, & \mathbf{r} \notin D_1,\\ \rho_i, & \mathbf{r} \in D_1,\end{cases} \tag{1}$$

where

$$L_U \equiv \frac{\partial}{\partial t}\Delta \mathbf{U} + \frac{g}{\rho_0}\left[\mathbf{k}\Delta\rho - \nabla\frac{\partial\rho}{\partial z}\right]; \quad L_\rho \equiv \frac{\partial\rho}{\partial t} - \frac{\rho_0}{g}N^2 w; \quad \varphi(\rho, \mathbf{U}) \equiv -\mathbf{U}\nabla\rho;$$
$$Q(\mathbf{U}) \equiv -\operatorname{curl}\operatorname{curl} t\,[(\mathbf{U}\cdot\nabla)\mathbf{U}].$$

The solution of the system (1) can be represented by the sum of two components, one of which describes the problem of the collapse of the intrusion zone in a stratified fluid, and the other the interaction of background IW with this zone. The collapse problem has been investigated well (see, e.g., [2]). It is known [3] that the solution of the collapse problem with viscosity taken into account for large times (the third stage of collapse) is a density field inhomogeneity in the form of a spot of mixed fluid that exists sufficiently long, dissipates extremely slowly at the level of its density. We assume that the geometric size of the domain D_1 and the degree of fluid mixing in it are such that the concluding stage of collapse sets in sufficiently rapidly. Then, following [1], we consider that the domain D that occurs is a density field inhomogeneity that does not change with time and is at rest. Consequently, the problem of background IW interaction with the domain D can be considered as a background IW scattering problem by inhomogeneities of the density field ρ_{i0} with initial conditions. Its solution can also be represented in the form of the sum of two components, one of which described the unperturbed IW field (we consider it known), and the other the intrinsically scattered field characterized by the velocity $U_S(r, t)$ and the density $\rho_S(r, t)$, where $U_S|_{t=0} = 0$ and $\rho_S|_{t=0} = 0$. As in [1], we limit ourselves in this paper to a single scattering approximation (Born approximation) within whose framework U_S and ρ_S satisfy the boundary value problem

$$L_U\{\rho_S, \mathbf{U}_S\} = 0, \quad L_S\{\rho_S, \mathbf{U}_S\} = \varphi(\rho_{i0}, \mathbf{U}_\Phi), \quad \rho_S, \mathbf{U}_S|_{|\mathbf{r}|\to} \to 0. \tag{2}$$

Leningrad. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 89-93, March-April, 1987. Original article submitted February 19, 1986.

 0021-8944/87/2802-0246$12.50